Construction of General Forms of Second Order Ordinary Differential Equations with known Solutions

by
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Abstract

A general form of linear second order ordinary differential equations transforming into equations with known solutions by a substitution with two arbitrary functions is pointed out.

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I. Installation of the problem and creation of the basic equations.

Let us consider the equation

$$v'' + p(x)v' + q(x)v = 0. \tag{1.1}$$

If we take

$$y = ve^{\frac{1}{2}\int p(x)dx} \tag{1.2}$$

we have

$$y'' + I(x)y = 0, \tag{1.3}$$

where

$$I(x) = q(x) - \frac{1}{4}\phi^2(x) - \frac{1}{2}\phi'(x) \tag{1.4}$$

The function $I(x)$ is usually called the "invariant of equation (1.1)".

We shall always assume that equation (1.1) has been transformed into equation (1.4).

I propose to introduce new variables defined by the relations

$$y = \psi(x)z, \tag{1.5}$$

$$u = \int \phi(x)dx \tag{1.6}$$

Equation (1.3) in the new variables will be

$$\frac{d^2z}{du^2} + \frac{\psi\phi' + 2\psi'\phi}{\psi\phi^2}\frac{dz}{du} + \frac{\psi'' + I\psi}{\psi\phi^2}z = 0 \tag{1.7}$$

The requiment that the first derivative must vanish gives the following relation between the functions $\phi(x)$ and $\psi(x)$:

$$\frac{\phi'}{\phi} + \frac{2\psi'}{\psi} = 0 \tag{1.8}$$

or

$$\phi(x) = \frac{c}{\psi^2(x)}, \tag{1.9}$$

$$u(x) = c\int \frac{dx}{\psi^2(x)}, \tag{1.10}$$

where c is arbitrary constant.

Equation (1.7) may now be written as

$$\frac{d^2z}{du^2} + \frac{1}{c^2}(\psi'' + I\psi)\psi^3 z = 0 \tag{1.11}$$

or

$$\frac{d^2z}{du^2} + \widetilde{I}(x)z = 0, \tag{1.12}$$

where

$$\widetilde{I}(u) = \frac{1}{c^2}(\psi'' + I\psi)\psi^3 \tag{1.13}$$

Using (1.13) and (1.10) we have the following relation between $I(x)$ and $\widetilde{I}$:

$$I(x) = c^2 \frac{\widetilde{I}(c\int \frac{dx}{\psi^2(x)})}{\psi^4} - \frac{\psi''}{\psi} \tag{1.14}$$

As a result we have that if the function $I(x)$ of the equation (1.3) is given by formula (1.14) then equation (1.3) is transformed to the form (1.12) by the substitutions (1.5) and (1.10).

There are many second order differential equations integrable in quadratures and many equation with well-known properties of theirs solutions. Taking $I(x)$ as the invariant of the equation mentioned above and setting $\psi(x)$ be arbitrary we shall obtain some families of equations which can be translated to known ones. The solutions of these equations will also be obtained.

Equation (1.3) satisfying the given conditions may be written on the form

$$\frac{d^2y}{dx^2} + [c^2 \frac{\widetilde{I}(c\int \frac{dx}{\psi^2(x)})}{\psi^4} - \frac{\psi''}{\psi}]y = 0, \tag{1.15}$$

and all solutions of this equation may be represented as

$$y = \psi(x)z(c\int \frac{dx}{\psi^2(x)}). \tag{1.16}$$

It is obvious that the solution of equation (1.1) may be written as

$$\nu = e^{-\frac{1}{2}\int p(x)dx}\psi(x)z(c\int \frac{dx}{\psi^2(x)}). \tag{1.17}$$

II. The simplest case.

Let us now consider the simplest case when equation (1.3) reduces to

$$\frac{d^2z}{dx^2} = 0. \tag{2.1}$$

Substituting $\widetilde{I}(x) \equiv 0$ into (1.14) we have

$$I = -\frac{\psi''}{\psi} \tag{2.2}$$

and all equations which may be transformed to the form (2.1) are described by the formula

$$\frac{d^2y}{dx^2} - \frac{\psi''}{\psi}y = 0 \tag{2.3}$$

The general solution of (2.3) have the form $z = c_1u + c_2$, where c_1 and c_2 are arbitrary constants, and u is a parametrical solution of (2.3). Hence, the general solution of (2.3) may be written on the form

$$y = \psi(c_1 \int \frac{dx}{\psi^2(x)} + c_2) \tag{2.4}$$

($c_1 \cdot c$ is changed to a new c_1).

Examples.
II.1. Let $\psi(x) = x^m$. Then (2.2) implies

$$I(x) = -m(m-1)x^{-2}$$

and equation (1.3) is Euler's equation:

$$\frac{d^2y}{dx^2} - \frac{m(m-1)}{x^2}y = 0$$

According to relation (2.4) we can write

$$\begin{aligned} y(x) &= x^m(c_1 \int \frac{dx}{x^{2m}} + c_2) \\ &= c_1x^{1-m} + c_2x^m, \quad \text{if} \quad m \neq 1/2 \\ &= \sqrt{x}(c_1 \ln x + c_2), \quad \text{if} \quad m = 1/2. \end{aligned}$$

II.2. Let $\psi(x) = x^m e^{kx^n}$, then

$$I(x) = -(m(m-1) + kn(2m+n-1)x^n + k^2n^2x^{2n})x^{-2}$$

and the appropriate equation of the form of (2.1) will be the following

$$x^2\frac{d^2y}{dx^2} - (m(m-1) + kn(2m+n-1)x^n + k^2n^2x^{2n})x^{-2}y = 0 \tag{2.6}$$

Its general solution is

$$y = x^{2m} e^{kx^n} (c_1 \int x^{-2m} e^{2kx^n} dx + c_2). \qquad (2.7)$$

It should be noted that equation (2.6) is a particular case of Lommel's equation [1]

$$x^2 \frac{d^2 y}{dx^2} + (bx^m + d)y = 0$$

The general solution of this equation is expressed by Bessel functions of index $p = \frac{1}{m}\sqrt{1-4d^2}$. In our case $p = 1/2$ and the Bessel functions are expressed as elementary functions.

Let $m = 0$, $n = 2$, $k = -1/2$. We have $\psi = e^{-\frac{x^2}{2}}$ and the equation is of the form

$$\frac{d^2 y}{dx^2} + (1 - x^2)y = 0$$

with general solution

$$y = e^{-\frac{x^2}{2}} (c_1 e^{x^2} + c_2).$$

In the case of $m = 0$, $n = 2$, $k = 1/2$ we have the equation

$$\frac{d^2 y}{dx^2} - (1 + x^2)y = 0$$

and general solution

$$\begin{aligned} y &= e^{\frac{x^2}{2}} (c_1 \int e^{-x^2} dx + c_2) \\ &= e^{\frac{x^2}{2}} (\frac{2\tilde{c}_1}{\sqrt{\pi}} \int_0^x e^{-t^2} dt + \tilde{c}_2) \\ &= e^{\frac{x^2}{2}} (\tilde{c}_1 \Phi(x) + \tilde{c}_2), \end{aligned}$$

where $\Phi(x)$ is the well-known error integral.

It is very interesting to find equations with constant coefficients which are transformed to the equation (2.1). To solve this problem let us assume

$$\psi(x) = \begin{cases} \sin mx \\ \cos mx. \end{cases}$$

We have in this case $I(x) = \pm m^2$, and the differential equation

$$\frac{d^2 y}{dx^2} \pm m^2 y = 0 \qquad (2.10)$$

The general solution of this equation is the following

$$y = \begin{cases} c_1 \cos mx + c_2 \sin mx, & \text{if we have} + m^2 \quad \text{in} \quad (2.10) \\ c_1 chmx + c_2 shmx, & \text{if we have} - m^2 \quad \text{in} \quad (2.10) \end{cases}$$

III. The case where equation (1.3) is transformed to an equation with constant coefficients.

Let the transformed equation be

$$\frac{d^2 z}{du^2} \pm a^2 z = 0, \tag{3.1}$$

where a =const. According to (1.14) we have

$$I(x) = \pm c^2 \frac{a^2}{\psi^4} - \frac{\psi''}{\psi} \tag{3.2}$$

and the appropriate equation is

$$\frac{d^2 y}{dx^2} + (\pm \frac{\beta^2}{\psi^4} - \frac{\psi''}{\psi})y = 0 \tag{3.3}$$

$(B^2 \doteq c^2 a^2)$.

Equation (3.2) is transformed to the equation

$$\frac{d^2 z}{du^2} = 0 \tag{2.1}$$

when $B = 0$.

The general solution of (3.3) is

$$y = \psi(x)(c_1 \cos(B \int \frac{dx}{\psi^2(x)}) + c_2 \sin(B \int \frac{dx}{\psi^2(x)}))$$

if the sign in front of B^2 is plus, and

$$y = \psi(x)(c_1 ch(B \int \frac{dx}{\psi^2(x)}) + c_2 \sin(B \int \frac{dx}{\psi^2(x)}))$$

in the case of a minus in front of B^2. Let us consider some cases.

III.1. $\psi(x) = x^m, \quad m \neq 1/2$. In this case (3.2) implies

$$I = \pm B^2 \cdot \frac{1}{x^{4m}} - m(m-1) \cdot \frac{1}{x^2}.$$

The equation has the form

$$x^2\frac{d^2y}{dx^2}+(\pm B^2x^{2(1-2m)}-m(m-1))y=0 \tag{3.4}$$

(this equation is also a particular case of Lommel's equation). And the general solution of the equation (3.4) has the form

$$y=x^m(c_1\cos(\frac{B}{1-2m}x^{1-2m})+C_2\sin(\frac{B}{1-2m}x^{1-2m}),$$

if the sign in front of B is plus, and

$$y=x^m(c_1ch(\frac{B}{1-2m}x^{1-2m})+C_2sh(\frac{B}{1-2m}x^{1-2m})),$$

if the sign in front of B is minus.

III. 2. Let $\psi(x)=\sqrt{x}$. As in the previous case we have

$$I=(\frac{1}{4}\pm B^2)\cdot\frac{1}{x^2}$$

and

$$\frac{d^2y}{dx^2}+\frac{m}{x^2}y=0, \tag{3.5}$$

where $m=\frac{1}{4}\pm B^2$. The equation (3.5) is Euler's equation.

III.3. $\psi(x)=\sqrt{ax^2+bx+c}$. We have according to (3.2)

$$I(x)=\frac{m}{(ax^2+bx+c)^2},$$

where $m=\frac{b^2-4ac}{4}\pm B^2$. The appropriate equation is

$$\frac{d^2y}{dx^2}+\frac{m}{(ax^2+bx+c)^2}y=0. \tag{3.6}$$

The representation of the general solution depeas on the sign of $\mathcal{D}=b^2-4ac$ and the sign in front of B.For example if $\mathcal{D}>0$ and $+B^2$ we get

$$y=\sqrt{ax^2+bx+c}\,(c_1\cos\sqrt{\frac{m}{b^2-4ac}-\frac{1}{4}}\ln|\frac{b+2ax-\sqrt{b^2-4ac}}{b+2ax+\sqrt{b^2-4ac}}|$$
$$+c_2\sin\sqrt{\frac{m}{b^2-4ac}-\frac{1}{4}}\ln|\frac{b+2ax-\sqrt{b^2-4ac}}{b+2ax+\sqrt{b^2-4ac}}|),$$

and if $-B^2$ we have

$$y=\sqrt{ax^2+bx+c}\,(c_1 ch\sqrt{\frac{1}{4}-\frac{m}{b^2-4ac}\ln|\frac{b+2ax-\sqrt{b^2-4ac}}{b+2ax+\sqrt{b^2-4ac}}|}$$
$$+c_2 sh\sqrt{\frac{1}{4}-\frac{m}{b^2-4ac}\ln|\frac{b+2ax-\sqrt{b^2-4ac}}{b+2ax+\sqrt{b^2-4ac}}|}).$$

It is only necessary to know the representation of the integral

$$\int\frac{dx}{ax^2+bx+c}$$

to write the general solutions in the cases $\mathcal{D}=0$ or $\mathcal{D}<0$.

III.4. Let us consider the homogenues Boussinesque's differential equation [1]

$$\frac{d^2y}{dx^2}+\frac{a^2}{(1+b(x-c)^2)^2}y=0 \tag{3.7}$$

Equation (3.7) is a particular case of equation (3.6). We have according to the remark above

$$y=\sqrt{1+b(x-c)}(c_1\cos(\sqrt{\frac{a^2}{b}+1}\mathrm{arctg}(\sqrt{b}(x-c)))+c_2\sin(\sqrt{\frac{a^2}{b}+1}\mathrm{arctg}(\sqrt{b}(x-c))).$$

III.5 Let us consider the homogeniuos Stokes equations of the form

$$\frac{d^2y}{dx^2}+\frac{a}{(bx-x^2)^2}y=0.$$

The general solution may be written on the form

$$y=\sqrt{bx-x^2}(c_1ch(\sqrt{\frac{1}{4}-\frac{a}{b^2}}\ln|\frac{x}{b-x}|)+c_2sh(\sqrt{\frac{1}{4}-\frac{a}{b^2}}\ln|\frac{x}{b-x}|))$$

III.5. $\psi(x)=\sqrt{\sec x}$ implies the following invariant and equation

$$I=\pm B^2\cos^2 x-\frac{3}{4}\mathrm{tg}^2 x-\frac{1}{2},$$

$$\frac{d^2y}{dx^2}+(\pm B^2\cos^2 x-\frac{3}{4}\mathrm{tg}^2 x-\frac{1}{2})y=0. \tag{3.8}$$

We have the solution

$$y=\sqrt{\sec x}(c_1\cos(B\sin x)+c_2\sin(B\sin x)),$$

if we have $+B^2$ in the equation (3.8) and

$$y = \sqrt{\sec x}(c_1 ch(B\sin x) + c_2 sh(B\sin x))$$

if we have $-B^2$.

III.6 $\psi(x) = \sqrt{xe^{-x}}$ implies the equation

$$4x^2\frac{d^2y}{dx^2} + (1 + 2x - x^2 \pm 4B^2e^{2x})y = 0.$$

In the case $+4B^2$ we have

$$y = \sqrt{xe^{-x}}(c_1\cos(BEi(x)) + c_2\sin(BEi(x)))$$

where

$$Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t}dt.$$

It is only nessecary to replace the functions sin and cos by the functions sh and ch to receive the general solution in the case $-4B^2$.

III.7. The choice of $\psi(x) = \sqrt{\ln x}$ leads to the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{(\ln x)^2}(\frac{1 + 2\ln x}{4x^2} \pm B^2)y = 0.$$

The solution of this equation again depends on the sign in front of B^2.

$$y = \begin{cases} \sqrt{\ln x}(c_1\cos(Bli(x)) + c_2\sin(Bli(x))), & \text{if sign "+"} \\ \sqrt{\ln x}(c_1 ch(Bli(x)) + c_2 sh(Bli(x))), & \text{if sign "}-\text{"} \end{cases}$$

where

$$li(x) = \int_{0}^{x} \frac{dt}{\ln t}.$$

IV. The case where equation (1.3) may be transformed to Euler's equation.

In this case the transformed equation is on the form

$$\frac{d^2z}{du^2} + \frac{\mathcal{D}}{u^2}z = 0, \tag{4.1}$$

where $\widetilde{I}(x) = \mathcal{D}/u^2$, $\mathcal{D}$ is constant. And we have using (1.14)

$$I(x) = \frac{\mathcal{D}}{\psi^4((\int \frac{dx}{\psi^2(x)})^2)} - \frac{\psi''}{\psi}. \tag{4.2}$$

The equation which may be transformed to the form (4.1) has the form

$$\frac{d^2y}{dx^2} + [\frac{\mathcal{D}}{\psi^4(\int \frac{dx}{\psi^2(x)})^2} -$$

It is known that the solution of equation (4.1) is the following

$$z = \begin{cases} \sqrt{u}(c_1 \cos(\sqrt{D - \frac{1}{4}} \ln u) + c_2 \sin(\sqrt{D - \frac{1}{4}} \ln u)) & \text{, if } D > \frac{1}{4} \\ \sqrt{u}(c_1 + c_2 \ln u) & \text{, if } D = \frac{1}{4} \\ \sqrt{u}(c_1 u^{\sqrt{\frac{1}{4} -}} + c_2 e^{-\sqrt{\frac{1}{4} - D}}) & \text{,if } D < \frac{1}{4}. \end{cases} \tag{4.3}$$

The general solution has the form

$$y = \begin{cases} \phi(x)\sqrt{\int \frac{dx}{\psi^2(x)}}((c_1 \cos(\sqrt{D - \frac{1}{4}} \ln \int \frac{dx}{\psi^2(x)}) + c_2 \sin(\sqrt{D - \frac{1}{4}} \ln \int \frac{dx}{\psi^2(x)})) & \text{,if } D > \frac{1}{4} \\ \psi(x)\sqrt{\int \frac{dx}{\psi^2(x)}}(c_1 + c_2 \ln \int \frac{dx}{\psi^2(x)}) & \text{,if } D = \frac{1}{4} \\ \psi(x)\sqrt{\int \frac{dx}{\psi^2(x)}}(c_1(\int \frac{dx}{\psi^2(x)})^{\sqrt{\frac{1}{4} - D}} + c_2(\int \frac{dx}{\psi^2(x)})^{-\sqrt{\frac{1}{4} - D}} & \text{,if } D < \frac{1}{4} \end{cases}$$

Examples:

IV.1. Setting $\psi = x^m$ we have according to (4.2)

$$I = \begin{cases} (D(1 - 2m)^2 - m(m-1))x^{-2} & \text{, if } m \neq \frac{1}{2} \\ (\frac{D}{(\ln x)^2} + \frac{1}{4})x^{-2} & \text{, if } m = \frac{1}{2} \end{cases}$$

and the associated equations

$$x^2 \frac{d^2y}{dx^2} + (\mathcal{D}(1 - 2m)^2 - m(m-1))y = 0 \quad \text{,if} \quad m \neq \frac{1}{2}$$
$$x^2 \frac{d^2y}{dx^2} + (\frac{D}{(\ln x)^2} + \frac{1}{4})y = 0 \qquad \text{,if} \quad m = \frac{1}{2}.$$

We easily see that the substitutions (1.5) and (1.6) by $m \neq \frac{1}{2}$ transformes equation (4.1) into equation of the same form.

If $m = \frac{1}{2}$, we have the solution

$$y = \begin{cases} \sqrt{x \ln x}(c_1 \cos(\sqrt{D - \frac{1}{4}} \ln \ln x) + c_2 \sin(\sqrt{D - \frac{1}{4}} \ln \ln x)) & \text{, if } D > \frac{1}{4} \\ \sqrt{x \ln x}(c_1 + c_2 \ln \ln x) & \text{, if } D = \frac{1}{4} \\ \sqrt{x \ln x}(c_1 (\ln x)^{\sqrt{\frac{1}{4} - D}} + c_2 (\ln x)^{-\sqrt{\frac{1}{4} - D}} & \text{, if } D < \frac{1}{4} \end{cases}$$

IV.2. Let us take $\psi = \sqrt{\sec x}$. Then

$$I(x) = D\mathrm{ctg}^2 x - \frac{3}{4}\mathrm{tg}^2 x - \frac{1}{2}$$

and

$$y = \begin{cases} \sqrt{\mathrm{tg}\, x}\,(c_1 \cos(\sqrt{D-\frac{1}{4}} \ln \sin x) + c_2 \sin(\sqrt{D-\frac{1}{4}} \ln \sin x)) & , \text{if } D > \frac{1}{4} \\ \sqrt{\mathrm{tg}\, x}\,(c_1 + c_2 \ln \sin x) & , \text{if } D = \frac{1}{4} \\ \sqrt{\mathrm{tg}\, x}\,(c_1 (\sin x)^{\sqrt{\frac{1}{4}-D}} + c_2 (\sin x)^{\sqrt{\frac{1}{4}-D}}) & , \text{if } D < \frac{1}{4} \end{cases}$$

V. The case where equation (1.3) may be transformed to Bessel's equation.

Bessel's equation is the following

$$u^2 \frac{d^2 w}{du^2} + u\frac{dw}{du} + (n^2 u^2 - \nu^2) w = 0$$

and has the normal form

$$\frac{d^2 z}{dx^2} + (n^2 - \frac{4\nu^2 - 1}{4u^2}) z = 0.$$

In this case we obtain (renaming the arbitrary constants)

$$\begin{aligned} \tilde{I}(u) &= a + bu^{-2}, \\ \tilde{I}(x) &= a + c(\int \frac{dx}{\psi^2(x)})^{-2}, \\ I(x) &= \frac{1}{\psi^4(x)}(B + \mathcal{D}(\int \frac{dx}{\psi^2(x)})^{-2})) - \frac{\psi''}{\psi}. \end{aligned}$$

The equations which may be transformed to Bessel's equation has the form

$$\frac{d^2 y}{dx^2} + (\frac{1}{\psi^4}(B + \mathcal{D}(\int \frac{dx}{\psi^2(x)})^{-2}) - \frac{\psi''}{\psi}) y = 0 \tag{4.4}$$

where B and $\mathcal{D}$ are arbitrary constants.

The general solution of (4.4) is the following

$$y = \psi(x)\sqrt{\int \frac{dx}{\psi^2(x)}}\, Z_{\frac{1}{2}\sqrt{1-4\mathcal{D}}}(\sqrt{B}\int \frac{dx}{\psi^2(x)}), \quad B \neq 0. \tag{4.5}$$

Z is the cylindrial function

$$Z_p(s) = C_1 I_p(s) + C_2 Y_p(s) \tag{4.6}$$

where the functions $I_p(s)$ and $Y_p(s)$ are determined in the following way

$$I_\nu(s)=\sum_{k=0}^{\infty}(-1)^k(\frac{s}{2})^{\nu+2k}/k!\Gamma(\nu+k+1) \qquad \forall s\in\mathbb{R}$$
$$Y_\nu(s)=(I_\nu(s)\cos\nu\pi-I_{-\nu}(s))/\sin\nu\pi \qquad ,\nu\notin\mathbb{Z},$$
$$Y_\nu(s)=\frac{2}{\pi}I_\nu(s)\ln\frac{s}{2}-\frac{1}{\pi}\sum_{k=1}^{\nu-1}\frac{(\nu-k-1)!}{k!}(\frac{2}{s})^{\nu-2k}-$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{s}{2})^{\nu+2k}}{k!(\nu+k)!}(\frac{\Gamma'(\nu+k+1)}{(\nu+k)!}+\frac{\Gamma'(k+1)}{k!}) \qquad ,\nu\in\mathbb{Z}.$$

It is known that if p equals one half an of odd number the solution of Bessel's equation may be finitely represented by elementary functions. It take place in our case when

$$\frac{1}{2}\sqrt{1-4D}=\frac{2n+1}{2}$$

that is

$$\mathcal{D}=-n(n+1), \quad \text{where} \quad n\in\mathbb{N}^+\cup 0.$$

Because of this property y is on elementary function of $S=\int dx/\psi^2(x)$.

IV.1. Let $\psi=x^m, \quad m\neq\frac{1}{2}$. In this case we have

$$I(x)=\frac{B}{x^{4m}}+\frac{\mathcal{D}(1-2m)^2-m(m-1)}{x^2}$$

and the equation is of the form

$$\frac{d^2y}{dx^2}+(\frac{B}{x^{4m}}+\frac{\mathcal{D}(1-2m)^2-m(m-1)}{x^2})y=0 \tag{4.7}$$

with general solution

$$y=\sqrt{x}Z_{\frac{\sqrt{1-4D(1-2m)-4m(m-1)}}{2(1-2m)}}(\frac{\sqrt{B}}{1-2m}x^{1-2m)} \tag{4.8}$$

IV.2. If $\psi=\sqrt{x}$ we have the equation

$$x^2\frac{d^2y}{dx}+(\frac{\mathcal{D}}{\ln^2 x}+L)y=0, \tag{4.9}$$

and the solution

$$y=\sqrt{x\ln x}Z_{\frac{1}{2}\sqrt{1-4\mathcal{D}}}(\sqrt{L-\frac{1}{4}\ln x}),$$

where $L=\mathcal{D}(1-2m)^2-m(m-1)$.

If $L = \frac{1}{4}$ the equation (4.8) may be transformed to Euler's equation.

IV.3. Let us look at Lommel's equation

$$x^2\nu'' + ax\nu' + (bx^k + c)\nu = 0, \quad k \neq 0 \tag{4.10}$$

Its invariant

$$I(x) = bx^{k-2} + (c - \frac{a^2}{4} + \frac{a}{2})x^{-2}$$

corresponds to the invariant of example IV.1. Substituting approppiate parameters into (4.8) we obtain the solution of Lommel's equation in the normal form

$$y = \sqrt{x} Z_{\frac{\sqrt{(1-a)^2-4c}}{k}}(\frac{2\sqrt{b}}{k}x^{\frac{k}{2}})$$

Using (1.17) we obtain the general solution of the equation (4.10)

$$\nu = x^{\frac{1-a}{2}} Z_{\frac{\sqrt{(1-a)^2-4c}}{k}}(\frac{2\sqrt{b}}{k}x^{\frac{k}{2}})$$

V. The case where equation (1.3) may be transformed to the Gauss (hypergeometric) equation.

This is the Gauss differential equation

$$u(u-1)\frac{d^2w}{du^2} + ((\alpha+\beta+1)u - \gamma)\frac{dw}{du} + \alpha\beta w = 0. \tag{5.1}$$

Its invariant is

$$\widetilde{I}(u) = \frac{1-\lambda^2}{4u^2} + \frac{1-\mu^2}{4(u-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{4u(u-1)},$$

where

$$\lambda = \gamma - 1, \quad \mu = \alpha + \beta - \gamma, \quad \nu = \alpha - \beta.$$

It is easy to obtain

$$\begin{aligned} I(x) = \frac{1}{\psi^4}(&\frac{1-\lambda^2}{4} \cdot \frac{1}{(\int \frac{dx}{\psi^2(x)})^2} + \frac{1-\mu^2}{4} \cdot \frac{1}{(\int \frac{dx}{\psi^2(x)} - D)^2} \\ &+ \frac{\lambda^2+\mu^2-\nu^2-1}{4} \cdot \frac{1}{\int \frac{dx}{\psi^2(x)}(\int \frac{dx}{\psi^2(x)} - \mathcal{D})} - \frac{\psi''}{\psi} \end{aligned} \tag{5.2}$$

The equation

$$\frac{d^2y}{dx^2}+(\frac{1}{\psi^4}(\frac{E}{(\int\frac{dx}{\psi(x)})^2}++\frac{F}{(\int\frac{dx}{\psi^2(x)}-D)^2}+\frac{G}{\int\frac{dx}{\psi^2(x)}\cdot(\int\frac{dx}{\psi^2(x)}-\mathcal{D})}-\frac{\psi''}{\psi})y=0 \quad (5.3)$$

is the image of the equation (5.1) under the transformation (1.5). The solution of the equation (5.1) is

$$w=c_1F(\alpha,\beta,\gamma,u)+c_2u^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,u) \quad (5.4)$$

in the case $\gamma\in\mathbb{N}$, where $F(\alpha,\beta,\gamma,u)$ is the well-known hypergeometric function wich may be represented as the hypergeometric series

$$F(\alpha,\beta,\gamma,u)=1+\frac{\alpha\beta_u}{1\cdot\gamma}u+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma(\gamma+1)}u^2+...$$

conveges for $|x|<1$. I think that the case of $\gamma\notin\mathbb{N}$ may be left to the reader!

The solution of the Gauss equation in normal form with the invariant $\widetilde{I}(u)$ as above is the following

$$z=\sqrt{(u-1)^{\alpha+\beta-\gamma+1}u^\gamma}(c_1F(\alpha,\beta,\gamma,u)+c_2u^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,u).$$

The solution of equation (5.3) may be written

$$y=\psi\sqrt{(\frac{dx}{\psi^2(x)})^\gamma(\int\frac{dx}{\psi^2(x)}-D)^{\alpha+\beta-\gamma+1}}(c_1F(\alpha,\beta,\gamma,\frac{1}{D}\int\frac{dx}{\psi^2(x)}))+$$
$$+c_2(\int\frac{dx}{\psi^2(x)})^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\frac{1}{D}\int\frac{dx}{\psi^2(x)})),$$

where

$$\alpha=\frac{1}{2}(1+\sqrt{1-4E}+\sqrt{1-4F}-\sqrt{1-4(E+F+G)},$$
$$\beta=\frac{1}{2}[1+\sqrt{1-4E}+\sqrt{1-4F}-\sqrt{1-4(E+F+G)},$$
$$\gamma=1+\sqrt{1-4E},$$

$\mathcal{D}$, E, F and G are arbitrary parameters of equation (5.3), c_1 and c_2 are arbitrary constants of integration.

Let us consider the particular cases as usuall. In the case of $\psi=x^m$ where $m\neq\frac{1}{2}$ we have

$$I(x)=(\frac{(1-2m)^2}{4}(\frac{1-\mu^2}{(1-D(1-2m)x^{2m-1})^2}+\frac{\lambda^2+\mu^2+\nu^2-1}{1-D(1-2m)x^{2m-1}})$$
$$+(\frac{1}{4}(1-2m)^2(1-\lambda^2)-m(m-1)))\cdot\frac{1}{x^2}$$

or

$$I(x) = (\frac{a}{(1-dx^k)^2} + \frac{b}{1-dx^k} + c)\frac{1}{x^2}$$

is short. The equation has the form

$$\frac{d^2y}{dx^2} + (\frac{a}{(1-dx^k)^2} + \frac{b}{1-dx^k})\frac{1}{x^2}y = 0$$

and its solution is

$$y = \sqrt{(-\frac{1}{k})^{1+\alpha+\beta}x^{-k(\alpha+\beta)}(kx^k+1)^{1+\alpha+\beta-\gamma}c_1F(\alpha,\beta,\gamma,\frac{1}{dx^k})}$$
$$+ c_2(-kx^k)^{\gamma-1}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,\frac{1}{dx^k})),$$

where

$$\alpha = \frac{1}{2k}(k+\sqrt{1-4c}+\sqrt{k^2-4a}+\sqrt{1-4(a+b+c)},$$
$$\beta = \frac{1}{2k}(k+\sqrt{1-4c}+\sqrt{k^2-4a}-\sqrt{1-4(a+b+c)}$$

and

$$\gamma = \frac{1}{k}(k+\sqrt{1-4c}).$$

It should be noted that the confluent hypergeometric equation given by

$$\lambda = \pm 1 \quad \text{and} \quad \mu = \pm 1$$

is not integrable using hypergeometrical functions. This case will be under consideration in the next paragraph.

VI. The case where equation (1.3) may be transformed to the Legendre differential equation.

The Legendre equation has the form

$$(u^2-1)\frac{d^2w}{du^2} + 2u\frac{dw}{du} - n(n+1) - \frac{\mu^2}{1-u^2})w = 0 \tag{6.1}$$

or

$$\frac{d^2z}{du^2} - (\frac{n(n+1)}{u^2-1} + \frac{\mu^2-1}{(u^2-1)^2})z = 0 \tag{6.2}$$

after translation to the normal form. The invariant $I(x)$ of this equation is

$$I(x) = -\frac{1}{\psi^4}\left(\frac{n(n+1)}{(\int \frac{dx}{\psi^2(x)})^2 - D^2} + \frac{D^2(\mu^2-1)}{((\int \frac{dx}{\psi^2(x)})^2 - D^2)^2}\right) - \frac{\psi''}{\psi}$$

where D is an arbitrary parameter. The equations which can be transformed to the Legendre equat may be written as

$$\frac{d^2y}{dx^2} - \frac{1}{\psi^4(x)}\left(\frac{n(n+1)}{(\int \frac{dx}{\psi^2(x)})^2 - D^2} + \frac{D^2(\mu^2-1)}{((\int \frac{dx}{\psi^2(x)})^2 - D^2)^2} + \frac{\psi''}{\psi}\right)y = 0 \tag{6.3}$$

The solutions of (6.1) is often written in the following form [2]

$$w = K_n^{(\mu)}(u) \tag{6.4}$$

and the solution of the equation (6.2) is

$$z = \sqrt{u^2-1}K^{(\mu)}(u) \tag{6.5}$$

where $K_n^{(\mu)}$ is the Legendre functions.

The solution of (6.3) is

$$y = \psi(x)\sqrt{\frac{1}{D^2}\int(\frac{dx}{\psi^2(x)})^2 - 1}K_n^{(\mu)}(\frac{1}{D}\int\frac{dx}{\psi^2(x)})$$

where D is the parameter of the equation.

Setting $\psi(x) = x^m \quad (m \neq 1/2)$ we have after transformations of the equation (6.3)

$$x^2(1-f^2x^k)^2\frac{d^2y}{dx^2} + (p + qx^k + \frac{1}{4}(1-\frac{k^2}{4})f^2x^{2k})y = 0. \tag{6.6}$$

It was noted in the previous paragraph that the hypergeometric equation is not integrable in terms of hypergeometric functions in the case of

$$\lambda = \pm 1 \quad \text{and} \quad \mu = \pm 1.$$

It is not difficult to show that this equation can be transformed to the Legendre equation. Indeed, in this case the invariant is

$$\tilde{I}(u) = \frac{1-\nu^2}{4u(u-1)} \tag{6.7}$$

and the equation will be

$$\frac{d^2z}{dx^2} + \frac{1-\nu^2}{4u(u-1)}z = 0 \tag{6.8}$$

Making the transformation

$$\xi = 2u - 1$$

we obtain the equation

$$\frac{d^2 z}{d\xi^2} + \frac{1-\nu^2}{4(\xi^2-1)} z = 0 \tag{6.9}$$

which is the Legendre equation of the form (6.2) by $\mu = 1$. As far as the solution of equation (6.9) may be written on the form

$$z = \sqrt{\xi^2 - 1} K^{(1)}_{\frac{\nu-1}{2}}(\xi)$$

the solution of the confluent hypergeometric equation (6.8) will be

$$z = 2\sqrt{u(u+1)} K^{(1)}_{\frac{\nu-1}{2}}(2u-1).$$

According to (1.14) and (6.7) we obtain

$$I(x) = \frac{1-\nu^2}{4\psi^4 \int \frac{dx}{\psi^2(x)}(\int \frac{dx}{\psi^2(x)}) - D)} - \frac{\psi''}{\psi},$$

and the equation may be written on the form

$$\frac{d^2 y}{dx^2} + (\frac{1-\nu^2}{4\psi^4(x)\int \frac{dx}{\psi^2(x)}(\int \frac{dx}{\psi^2(x)} - D)} - \frac{\psi''}{\psi})y = 0. \tag{6.10}$$

The solution of this equation is

$$y = \frac{2\psi(x)}{D}\sqrt{\int \frac{dx}{\psi^2(x)}(\int \frac{dx}{\psi^2(x)} + D)} K^{(1)}_{\frac{\nu-1}{2}}(\frac{2}{D}\int \frac{dx}{\psi^2(x)} - 1)$$

where $K^{(1)}_{\frac{\nu-1}{2}}(S)$ is an appropriate Legendre function and D is the parameter of equation. Making $\psi = x^m (m \neq 1/2)$ in (6.1) and transforming we obtain

$$4x^2(1+fx^k)\frac{d^2 y}{dx^2} + (b - f(k^2-1)x^k)y = 0. \tag{6.11}$$

The equations (6.11) and (6.6) are Euler's equations when $m = 1/2$.

VII. The case where equation (1.1) may be transformed to Whittaker's equation.

This equation has the form

$$4u^2 \frac{d^2 z}{du^2} - (u^2 - 4\lambda u + 4\mu^2 - 1)z = 0 \tag{7.1}$$

We have

$$\tilde{I}(u) = \frac{1-4\mu^2}{4u^2} + \frac{\lambda}{u} - \frac{1}{4},$$
$$I(x) = \frac{1}{\psi^4(x)}(\frac{1-4\mu^2}{4} \cdot \frac{1}{(\int \frac{dx}{\psi^2(x)})^2} + \frac{\lambda c}{\int \frac{dx}{\psi^2(x)}} - \frac{c^2}{4}) - \frac{\psi''}{\psi},$$

and the equations which may be transformed to the Whittaker equation is written in the form

$$\frac{d^2y}{dx^2} + (\frac{1}{\psi^4(x)}(\frac{1-4\mu^2}{4} \cdot \frac{1}{(\int \frac{dx}{\psi^2(x)})^2} + \frac{\lambda c}{\int \frac{dx}{\psi^2(x)}} - \frac{c^2}{4}) - \frac{\psi''}{\psi})y = 0 \tag{7.2}$$

The equation (7.2) is transformed to Bessel's equation by $\lambda = 0$ and to Euler's equation by $c = 0$.

The solution of equation (7.1) is [1] $z = c_1 M_{\lambda,\mu}(u) + c_2 M_{\lambda,-\mu}(u)$ when $2\mu \notin \mathbb{Z}$, and $z = c_1 W_{\lambda,\mu}(u) + c_2 W{-\lambda,\mu}(-u)$ when $2\mu \in Z$. The solution of equation (7.2) is in the appropriate cases

$$y = \psi(x)(c_1 M_{\lambda,\mu}(c\int \frac{dx}{\psi^2(x)}) + c_2 M_{\lambda,-\mu}(c\int \frac{dx}{\psi^2(x)})),$$
$$y = \psi(x)(c_1 W_{\lambda,\mu}(c\int \frac{dx}{\psi^2(x)}) + c_2 W_{-\lambda,\mu}(-c\int \frac{dx}{\psi^2(x)}))$$

where as usual c_1 and c_2 are arbitrary constants, c is the parameter of the equation (7.2). For information of the function $M_{k,e}(S)$ and $W_{k,e}(S)$ see [2].

VII.1. Set $\psi = x^m (m \neq 1/2)$. In this case the equation receives the form

$$\frac{d^2y}{dx^2} - (\frac{a^2}{x^{4m}} + \frac{ab}{x^{2m+1}} + \frac{c}{x^2})y = 0 \tag{7.3}$$

after some transformations. Making the substitution

$$s = 1 - 2m$$

we have

$$x^2 \frac{d^2y}{dx^2} - (a^2 x^{25} + abx^5 + c)y = 0 \tag{7.4}$$

The solution of equation (7.3) is

$$y = x^m (c_1 M_{\frac{b}{2(2m-1)}, \frac{\sqrt{1+4c}}{2(2m-1)}} (\frac{2a}{1-2m} x^{1-2m})$$
$$+ c_2 M_{\frac{b}{2(2m-1)}, \frac{\sqrt{1+4c}}{2(2m-1)}} (\frac{2a}{1-2m} x^{1-2m})$$

and the solution of equation (7.4) is

$$y = x^{\frac{1-s}{2}} (c_1 M_{-\frac{b}{2s}, \frac{\sqrt{1+4c}}{2s}} (\frac{2a}{s} x^s)$$
$$+ c_2 M_{-\frac{b}{2s}, \frac{-\sqrt{1+4c}}{2s}} (\frac{2a}{s} x^s)).$$

In the case where

$$2\mu = \frac{\sqrt{1+4c}}{2m-1}$$

or

$$2\mu = \frac{\sqrt{1+4c}}{s}$$

is an integer the solution must be written using the functions $W_{k,e}(u)$ and $W_{k,e}(-u)$.

Setting $s = 1$ in equation (7.4) we receive so called radial-wave's equation [1]

$$x^2 \frac{d^2 y}{dx^2} - (a^2 x^2 + abx + c) y = 0$$

with the solution

$$y = c_1 M_{-\frac{b}{2}, \frac{\sqrt{1+4c}}{2}} (2ax) + c_2 M_{-\frac{b}{2}, -\frac{\sqrt{1+4c}}{2}} (2ax).$$

Setting $a = \beta, b = 2\alpha, c = \alpha^2 - \alpha$ we have the equation

$$x^2 \frac{d^2 y}{dx^2} - ((\beta x + \alpha)^2) - \alpha) y = 0$$

whose solution is

$$y = c_1 M_{-\alpha, \alpha - \frac{1}{2}} (2\beta x) + c_2 M_{-\alpha, \frac{1}{2} - \alpha} (2\beta x)$$

The solution of the last equation may be written in the form (see [1])

$$y = x^\alpha e^{\beta x} (c_1 + c_2 \int x^{-2\alpha} e^{-2\beta x} dx).$$

Another interesting case we get by letting

$$a = 2D, b = -\frac{2n+1}{2}, c = -\frac{3}{16}.$$

This is the equation

$$x^2 \frac{d^2y}{dx^2} - (4\mathcal{D}^2x^2 - \mathcal{D}(2n+1)x - \frac{3}{16})y = 0 \tag{7.5}$$

The solution of this equation is

$$y = c_1 M_{\frac{2n+1}{4}, \frac{1}{4}}(4\mathcal{D}x) + c_2 M_{\frac{2n+1}{4}, -\frac{1}{4}}(4\mathcal{D}x) \tag{7.6}$$

and the solution may be expressed in terms of Hermite polynomials or integral of probability. We shall term to this equation later.

In the case of

$$s = 2, a = 2 \text{ and } c = 0$$

we have the Weber differential equation in normal form

$$\frac{d^2y}{dx^2} - (x^2 + b)y = 0 \tag{7.7}$$

The Scrödinger wave equation is reduced to this equation in the case of harmonic oscillater. The solution of equation (7.7) is the following

$$y = x^{-\frac{1}{2}}(c_1 M_{-\frac{b}{4}, \frac{1}{4}}(x^2) + c_2 M_{-\frac{b}{4}, -\frac{1}{4}}(x^2))$$

The particular case of the Weber equation we have when

$$s = 2, a = 1/2, b = d/2, c = 0$$

is

$$4\frac{d^2y}{dx^2} - (x^2 + d)y = 0 \tag{7.8}$$

The solution of this equation is

$$y = x^{-\frac{1}{2}}(c_1 M_{-\frac{d}{8}, \frac{1}{4}}(\frac{1}{2}x^2) + c_2 M_{-\frac{d}{8}, -\frac{1}{4}}(\frac{1}{2}x^2)).$$

VII.2. Let $4 = \sqrt{x}$. We have

$$I(x) = (\frac{1 - 4\mu^2}{4} \frac{1}{(lnx)^2} + \lambda c \frac{1}{\ln x} + \frac{1 - c^2}{4}) \cdot \frac{1}{x^2}$$

and the equation will be

$$x^2\frac{d^2y}{dx^2}+(\frac{1-4\mu^2}{4}\cdot\frac{1}{(\ln x)^2}+\lambda c\frac{1}{\ln x}+\frac{1-c^2}{4})\frac{1}{x^2}y=0$$

with the general solution

$$y=\sqrt{x}(c_1M_{\lambda,\mu}(c\ln x)+c_2M_{\lambda,-\mu}(c\ln x)).$$

VII.3. Let us assume $\psi=x^{\frac{1-k}{x^2}}e^{kx^n}$. Proposing in (7.2)

$$s=-2k$$

we receive the equation

$$x^2\frac{d^2y}{dx^2}-((\mu^2s^2n^2-\lambda sne^{sx^n}+\frac{c^2}{4}e^{2sx^n})x^{2n}+\frac{n^2-1}{4})y=0$$

with the solution

$$y=x^{\frac{1-n}{2}}e^{-\frac{s}{2}x^n}(c_1M_{\lambda,\mu}(\frac{c}{sn}e^{sx^n})$$
$$+c_2M_{\lambda,-\mu}(\frac{c}{sn}e^{sx^n})).$$

VIII. The case where equation (1.3) may be transformed to the Mathieu and Lame equations.

VIII.A. The Mathieu equation has the form

$$\frac{d^2z}{du^2}+(a-2g\cos 2u)z=0 \tag{8.1}$$

where a and g are constants. In this case

$$\tilde{I}(u)=a-2g\cos 2u$$

or

$$\tilde{I}(u)=a-2g+4gsin^2u$$

and the function $I(x)$ may be written in the form

$$I(x)=\frac{1}{\psi^4(x)}(C\sin^2(\int c\frac{dx}{\psi^2(x)})+b)-\frac{\psi''}{\psi}. \tag{8.2}$$

The equation which may be transformed to the Mathieu equation is the following

$$\frac{d^2y}{dx^2}+(\frac{1}{\psi^4(x)}(C\sin^2(c\int\frac{dx}{\psi^2(x)})+b)-\frac{\psi''}{\psi})y=0 \tag{8.3}$$

It is known [3], that the solution of the equation (8.1) is

$$z = c_1 ce_\nu(u,g) + c_2 se_\nu(u,g) \tag{8.4}$$

in the case where $\nu = \sqrt{a}$ is not an integer. In this case the solution of equation (8.3) is in the form

$$y = \psi(x)(c_1 ce_{\frac{1}{c}\sqrt{b+\frac{C}{2}}}(c\int\frac{dx}{\psi^2(x)}, \frac{C}{4c^2})$$
$$+ c_2 se_{\frac{1}{c}\sqrt{b+\frac{C}{2}}}(c\int\frac{dx}{\psi^2(x)}, \frac{C}{4c^2})).$$

c, C, b and g are parameters of the equation, c_1 and c_2 are arbitrary constants.

The properties of the functions se and ce see [2]. Setting $\sqrt[4]{1-x^2} = \psi$ we have according to (8.2)

$$I(x) = \frac{1}{1-x^2}(c\sin^2(C\arcsin \mathrm{x}) + \mathrm{b}) + \frac{2+\mathrm{x}^2}{4(1-\mathrm{x}^2)^2},$$

if $C = 1$

$$I(x) = \frac{b+cx^2}{1-x^2} + \frac{2+x^2}{4(1-x^2)^2},$$

and the equation has the form

$$\frac{d^2y}{dx^2} + (\frac{b+cx^2}{1-x^2} + \frac{2+x^2}{4(1-x^2)^2})y = 0.$$

The solution of last equation will be

$$y = \sqrt[4]{1-x^2}(c_1 ce_{\sqrt{b+\frac{c}{2}}}(arc\sin x, \frac{c}{4}) +$$
$$+ c_2 se_{\sqrt{b+\frac{c}{2}}}(arc\sin x, \frac{c}{4})).$$

VIII.B. Changing $u \to iu$ in equation (8.1) we have the equation

$$\frac{d^2z}{du^2} - (a - 2gch2u)z = 0 \tag{8.5}$$

where a and g are constants. In this case

$$\tilde{I}(u) = 2gch2u - a = 4gsh^2u - 2g - a$$

and

$$I(x) = \frac{1}{\psi^4(x)}(Csh^2(c\int\frac{dx}{\psi^2(x)}) - b) - \frac{\psi''}{\psi}.$$

The equation

$$\frac{d^2y}{dx^2} + \frac{1}{\psi^4(x)}(Csh^2(c\int\frac{dx}{\psi^2(x)} - b) - \frac{\psi''}{\psi})y = 0 \tag{8.6}$$

may be transformed to the Mathieu equation. The solution of the equation (8.5) may be written using the modified Mathieu functions

$$z = c_1 Ce_\nu(u, g) + c_2 Se_\nu(u, g),$$

and the solution of (8.6) will be

$$y = \psi(x)(c_1 Ce_{\frac{1}{c}\sqrt{b+\frac{c}{2}}}(c\int\frac{dx}{\psi^2(x)}, \frac{C}{4c^2})$$
$$+ c_2 Se_{\frac{1}{c}\sqrt{b+\frac{c}{2}}}(c\int\frac{dx}{\psi^2(x)}, \frac{C}{4c^2})).$$

As in the previous example we have if

$$\psi = \sqrt[4]{1+x^2}$$

that $\int\frac{dx}{\psi^2(x)} = \int\frac{dx}{\sqrt{1+x^2}} = \mathrm{Arsh}\,\mathrm{x}$,

$$I(x) = \frac{1}{1+x^2}(Csh^2(c\mathrm{Arsh}\,\mathrm{x}) - \mathrm{b}) - \frac{2-\mathrm{x}^2}{4(1+\mathrm{x}^2)^2}.$$

If $c = 1$ we receive the equation

$$\frac{d^2y}{dx^2} - (\frac{b - cx^2}{1+x^2} + \frac{2-x^2}{4(1+x^2)})y = 0$$

and the solution may be written by analogic the example above.

VIII.C This is the Lame equation

$$\frac{d^2z}{du^2} + (gsn^2u + a)z = 0 \tag{8.7}$$

where g and a are constants, snu is determed as the inverse function of

$$u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \tag{8.8}$$

and depends on the parameter k.

It is easy to show that the equation

$$\frac{d^2y}{dx^2}+\frac{1}{\psi^4(x)}(Csn^2(c\int\frac{dx}{\psi^2(x)}+b)-\frac{\psi''}{\psi})y=0$$

may be translated to the Lame equation.

The construction of the solution of this equation is the same as in previous cases.

$$\frac{d^2z}{du^2}+(n+\frac{1}{2}-\frac{u^2}{4})z=0 \qquad (9.1)$$

where $n \in \mathbb{N}^+$. We receive after some calculations

$$I(x)=\frac{D}{\psi^4(x)}(2n+1-D(\int\frac{dx}{\psi^2(x)})^2)-\frac{\psi''}{\psi} \qquad (9.2),$$

where D is an arbitrary constant. Hence the equation

$$\frac{d^2y}{dx^2}+\frac{D}{\psi^4(x)}(2n+1-D(\int\frac{dx}{\psi^2(x)})^2-\frac{\psi''}{\psi})y=0 \qquad (9.3)$$

may be transformed to Weber's equation (9.1).

The solution of equation (9.1) may be represented using Hermite's polynomials $H_n(x)$

$$H_n(x)=(2x)^n-\frac{n(n-1)}{1!}(2x)^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}+\cdots$$

This solution is the following [2]

$$z=z_n=\frac{e^{-\frac{u^2}{4}}}{\sqrt{n!\sqrt{2\pi}}}H_n(u)$$

The functions $z_n(u)$ are called the functions of parabolic cylinder. They can be calculated using tables of derivatives of the probability integral $\Phi_n(x)$

$$z_n=c_n\frac{e^{-\frac{u^2}{4}}\cdot 2^{-n}\Phi_{n+1}(\frac{u}{\sqrt{2}})}{\Phi_1(\frac{u}{\sqrt{2}}}$$

where

$$c_n=(-1)^n\sqrt{\frac{2n}{n!\sqrt{2\pi}}}$$

Hence the solution of the equation (9.3) may be written

$$y=y_n=\frac{\psi(x)e^{-\frac{1}{2}D(\int\frac{dx}{\psi^2(x)})^2}H_n(\sqrt{2D}\int\frac{dx}{\psi^2(x)})}{\sqrt{n!\sqrt{2\pi}}} \qquad (D\neq 0)$$

or

$$y_n = \frac{\psi(x)c_n e^{-\frac{1}{2}D(\int \frac{dx}{\psi^2(x)})^2} \cdot 2^{-n}\Phi_{n+1}(\sqrt{D}\int \frac{dx}{\psi^2(x)})}{\Phi_1(\sqrt{D}\int \frac{dx}{\psi^2(x)})} \qquad (D \neq 0)$$

Setting $\psi = x^m$ $(m \neq \frac{1}{2})$ we receive the following invariant of the form (9.2)

$$I(x) = \frac{D(2n+1)^{4m}}{x} - \frac{D^2}{(1-2m)^2 x^{2(4m-1)}} - \frac{m(m-1)}{x^2}.$$

Denoting

$$s = 2(1-2m)$$

we have the equation

$$x^2\frac{d^2y}{dx^2} - (4(\frac{D}{s})^2 x^{2s} - D(2n+1)x^s - \frac{1}{4}(1-\frac{s^2}{4}))y = 0 \qquad (9.4)$$

with the solution of the form

$$y_n = \frac{x^{\frac{2-s}{4}} \cdot H_n(\frac{2\sqrt{2D}}{s} x^{\frac{s}{2}})}{\sqrt{n!\sqrt{2\pi}}\, e^{\frac{2d}{s^2}x^s}} \qquad (D \neq 0)$$

or on the form

$$y_n = \frac{c_n x^{\frac{2-s}{4}}}{2^n\, e^{\frac{2D}{s^2}x^s}} \cdot \frac{\Phi_{n+1}(\frac{2\sqrt{D}}{s}x^{\frac{s}{2}})}{\Phi_1(\frac{2\sqrt{D}}{s}x^{\frac{s}{2}})}. \qquad (D \neq 0)$$

Equation (9.4) is a particular case of the equation (7.4) where

$$a = \frac{2D}{s}, b = -(n+\frac{1}{2})s, c = \frac{1}{4}(\frac{s^2}{4} - 1).$$

Hence the solution of the equation (9.4) may be written using Whittaker's functions

$$y = x^{\frac{1-s}{2}}(c_1 W_{\frac{2n+1}{4},\frac{1}{4}}(\frac{4D}{s^2}x^s) + c_2 W_{-\frac{2n+1}{4},\frac{1}{4}}(-\frac{4D}{s^2}x^s)).$$

Setting $s = 1$ we have

$$x^2\frac{d^2y}{dx^2} - (4D^2 \cdot x^2 - D(2n+1)x - \frac{3}{16})y = 0 \qquad (9.5)$$

This is a particular case of the radial-wave's equation. The solution in this case may be written on the following equivalent form

$$y = \frac{x^{\frac{1}{4}}}{\sqrt{n!\sqrt{2\pi}}\, e^{2Dx}} \cdot H_n(2\sqrt{2Dx}),$$
$$y = \frac{c_n x^{\frac{1}{4}}}{2^n e^{2Dx}} \cdot \frac{\Phi_{n+1}(2\sqrt{Dx})}{\Phi_1(2\sqrt{Dx})},$$

and

$$y = c_1 W_{\frac{2n+1}{4},\frac{1}{4}}(4Dx) + c_2 W_{-\frac{2n+1}{4},\frac{1}{4}}(-4Dx).$$

If $n = 0$ the equation (9.5) will be of the following form

$$x^2 \frac{d^2y}{dx^2} - (4D^2x^2 - Dx - \frac{3}{16})y = 0$$

with the first form of the solution

$$y = \sqrt[4]{\frac{x}{2\pi}} e^{-2Dx} H_0(2\sqrt{2Dx}) = e^{-2Dx} \sqrt[4]{\frac{x}{2\pi}}$$

because $H_0 = 1$.

The second form of the solution of equation (9.5) is

$$y = c_0 \cdot e^{-2Dx} \sqrt[4]{x} = e^{-2Dx} \sqrt{\frac{x}{2\pi}}$$

because

$$c_n = (-1)^n \sqrt{\frac{2^n}{n!\sqrt{2\pi}}} \text{ and } c_0 = \frac{1}{\sqrt[4]{2\pi}}.$$

The third form of the solution is

$$y = c_1 W_{\frac{1}{4},\frac{1}{4}}(4Dx) + c_2 W_{-\frac{1}{4},\frac{1}{4}}(-4Dx).$$

X. The equations of the form

$$v'' + p(x)v' + g(x) = 0. \tag{1.1}$$

Transforming linear differential equation of the second order (1.1) to the normal form (1.3) by the substitution (1.2) we receive the invariant

$$I(x) = g(x) - \frac{1}{4}p^2(x) - \frac{1}{2}p'(x) \tag{1.4}$$

Let us go back. We have the given equation

$$y'' + I(x)y = 0 \tag{1.3}$$

and want to construct the most general linear differential equation of the form (1.1). Expressing the functions $g(x)$ in terms of $I(x)$ and $p(x)$ or $p(x)$ in terms of $I(x)$ and $g(x)$, we shall obtain an equation with two arbitrary functions: $I(x)$ and $p(x)$ or $I(x)$ and $g(x)$. For example according to (1.4) we have

$$g(x) = I(x) + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x).$$

Hence we receive

$$v'' + p(x)v' + (I(x) + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))y = 0 \tag{10.1}$$

As far as

$$I(x) = c^2 \cdot \frac{\tilde{I}(c\int\frac{dx}{\psi^2(x)})}{\psi^4(x)} - \frac{\psi''}{\psi}$$

we have

$$v'' + p(x)v' + (\frac{c^2\tilde{I}(c\int\frac{dx}{\psi^2(x)})}{\psi^4(x)} - \frac{\psi''(x)}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))y = 0 \tag{10.2}$$

Hence we have obtained some differential equations of general form which may be transformed to equation (1.3) by the substitutions (1.2), (1.5) and (1.10). The appropriate special functions in the representation of the solution of this equation depend on the form of the invariant

$$\tilde{I}(c\int\frac{dx}{\psi^2(x)})$$

The received equation includes two arbitrary functions (and an arbitrary constant of $\int\frac{dx}{\psi^2(x)}$). Choosing these functions we can receive a number of equations.

The solution of equation (1.1) may be represented in the form

$$v = ye^{-\frac{1}{2}\int p(x)dx}$$

e.g. expressed in the form of the solution of the same equations writting in normal form. We can write

$$v = e^{-\int p(x)dx}\psi(x)z(c\int\frac{dx}{\psi^2(x)}).$$

Let us finally list the equation transformed to the various examples considered above:

1. To $\frac{d^2z}{du^2} = 0$

$$v'' + p(x)v' - (\frac{\psi''(x)}{\psi(x)} - \frac{1}{4}p^2(x) - \frac{1}{2}p'(x))v = 0$$

2. To equation with constant coefficients

$$v'' + p(x)v' + (\pm\frac{B^2}{\psi^4(x)} - \frac{\psi''(x)}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))v = 0$$

3. To Euler's equation

$$v'' + p(x)v' + (\frac{D}{\psi^4(\int \frac{dx}{\psi^2(x)})^2} - \frac{\psi''(x)}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))v = 0$$

4. To Bessel's equation

$$v'' + p(x)v' + (\frac{1}{\psi^4}(B + D(\int \frac{dx}{\psi^2(x)})^{-2}) - \frac{\psi''}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))v = 0$$

5. To the Gauss equation

$$v'' + p(x)v' + (\frac{1}{\psi^4(x)}(\frac{E}{(\int \frac{dx}{\psi^2(x)})^2} + \frac{F}{[\int \frac{dx}{\psi^2(x)-D}]^2}$$
$$+ \frac{G}{\int \frac{dx}{\psi^2(x)}(\int \frac{dx}{\psi^2(x)} - D)}) - \frac{\psi''(x)}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))v = 0$$

7. To Whittaker's equation

$$v'' + p(x)v' + (\frac{1}{\psi^4(x)}(\frac{1-4\mu}{4(\int \frac{dx}{\psi^2(x)})^2} + \frac{\lambda c}{\int \frac{dx}{\psi^2(x)}} - \frac{c^2}{4})$$
$$- \frac{\psi''}{\psi} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))v = 0$$

8. To Mathieu equation

$$v' + p(x)v' + (\frac{1}{\psi^4(x)}(C\sin^2(c\int \frac{dx}{\psi^2(x)}) + b) - \frac{\psi''(x)}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))v = 0$$

9. To Lame's equation

$$v'' + p(x)v + (\frac{1}{\psi^4(x)}(Csn^2(c\int \frac{dx}{\psi^2(x)} + b))-$$
$$- \frac{\psi''(x)}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x))v = 0$$

10. To Hermite's equation or to the equation which is involves the integral of probability

$$v'' + p(x)v' + (\frac{D}{\psi^4(x)}(2n + 1 - D(\int \frac{dx}{\psi^2(x)})^2)$$
$$- \frac{\psi''(x)}{\psi(x)} + \frac{1}{2}p'(x) + \frac{1}{4}p^2(x))v = 0$$

I think that constants in these equations are self-explanatory of this stage.

XI. Riccati equation.

It is easy to prolong this paper by showing that Riccati equations are integrable in appropriate cases. I shall show briefly how to do it. Subsituting

$$Y(x) = -\frac{1}{a(x)}\frac{v'}{v} \tag{11.1}$$

into the Riccati equation

$$Y' = a(x)Y^2 + b(x)Y + e(x) \tag{11.2}$$

we receive a linear homogeneoussecond order ordinary differential equation (1.1) where

$$p(x) = -(b(x) + \frac{a'(x)}{a(x)}), q(x) = a(x)c(x) \tag{11.3}$$

Let us go back. Supposing $p(x), q(x)$ and $a(x)$ to be arbitrary functions we can find $b(x)$ and $c(x)$. Using (11.3) we have

$$b(x) = -(p(x) + \frac{a'(x)}{a(x)}), c(x) = \frac{q(x)}{a(x)} \tag{11.4}$$

and we can write equation (11.2) on the form of

$$\frac{dY}{dx} = a(x)Y^2 - (p(x) + \frac{a'(x)}{a(x)})Y + \frac{q(x)}{a(x)} \tag{11.5}$$

According to (1.4)

$$q(x) = I(x) + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x)$$

Using (1.14) we finally have the equation

$$\begin{aligned} Y' = a(x)Y^2 - (b(x) + \frac{a'(x)}{a(x)})Y + \frac{1}{a(x)}[\frac{c^2\tilde{I}(c\int \frac{dx}{\psi^2(x)})}{\psi^4(x)} \\ - \frac{\psi''(x)}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x)]. \end{aligned} \tag{11.6}$$

It is easy to obtain the solution of equation (11.6) on the form

$$Y = \frac{1}{a(x)}(\frac{b(x)}{2} - \frac{y'}{y}) \tag{11.7}$$

where y is the solution of equation (1.3) and on the form

$$Y = \frac{1}{a(x)}(\frac{b(x)}{2} - \frac{\psi'}{\psi} - \frac{z'}{z}) \tag{11.8}$$

where z is the solution of equation (1.12).

As an example let us obtain the general form of the Riccati equation which has a solution expressed by Bessel's functions. We have in this case

$$\tilde{I}(x) = a + c(\int \frac{dx}{\psi^2(x)})^{-2}$$

and according to (11.6)

$$\begin{aligned} Y' = a(x)Y^2 - (b(x) + \frac{a'(x)}{a(x)})Y + \frac{1}{a(x)}(\frac{1}{\psi^4(x)}(B + D(\int \frac{dx}{\psi^2(x)})^{-2}) \\ - \frac{\psi''}{\psi(x)} + \frac{1}{4}p^2(x) + \frac{1}{2}p'(x)). \end{aligned} \tag{11.8}$$

and its solution

$$\begin{aligned} Y = \frac{1}{a(x)}(\frac{b(x)}{2} - \frac{\psi'(x)}{\psi} - \frac{1}{2\psi^2(x)\int \frac{dx}{\psi^2(x)}} + \frac{\sqrt{B(1-4D)}}{2\psi^4(x)\int \frac{dx}{\psi^2(x)}} \\ - \frac{B}{\psi^4(x)} \frac{Z_{\frac{1}{2}\sqrt{1-4D}-1}(\sqrt{B}\int \frac{dx}{\psi^2(x)})}{Z_{\frac{1}{2}\sqrt{1-4D}}(\sqrt{B}\int \frac{dx}{\psi^2(x)})}) \end{aligned}$$

(see chapt. IV). It would be easy to find conditions of solving equation (11.8) by elementary functions and so on.

It should be noted that the main idea of this paper is influenced by [4].

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REFERENCES

[1] Kamke, E., 1942: Differentialgleichungen. Lösungsmetoden und Lösungen. B.1: Gewönliche Differentialgleichungen. Leipzig.

[2] Jahnke, E. & Emde, F., 1943: Tables of functions with formulas and curves. N.Y.

[3] McLachlen, N.W., 1951: Theory and application of Mathieu functions. Oxford.

[4] Manshalovskii, V.P., 1953: Integrability of some homogeneous linear second order ordinary differential equations with variable coefficients using special functions. Kharkov (in Russian).